11/12 Recitation Solutions

1. How many (nonreal) complex eigenvalues can a 9×9 matrix have?

   Since every matrix must have an even number of (nonreal) complex eigenvalues, a 9×9 matrix must have either $0, 2, 4, 6$, or 8 such eigenvalues.

2. Suppose a 2×2 matrix A has eigenvalue $\lambda = 2 - i$.

   (a) Is A invertible?

   Since $2-i$ is an eigenvalue of A, the other eigenvalue is $\overline{2-i} = 2+i$. Since neither of these are 0, A is invertible.

   (b) What is the determinant of A?

   Since $\det A$ is the product of the eigenvalues of A (HW8), $\det A = (2+i)(2-i) = 5$.

   (c) Is A symmetric?

   Since the eigenvalues of A are not real, A is not symmetric.

3. Recall that for any matrix A, A^TA is positive semi-definite. Is AA^T also positive semi-definite? If so, explain why. If not, provide a counterexample.

   Yes. Let $B = A^T$. Then $AA^T = B^T(B^T)^T = B^TB$ is positive semi-definite.

4. Let $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (a) Orthogonally diagonalize S.

   We first find the eigenvalues of S.

$$\begin{aligned}
\det(S-\lambda I) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\
&= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} \text{ (Cofactor expansion along R1)} \\
&= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(\lambda + 1) \\
&= -\lambda(\lambda - 1)(\lambda + 1) + 2(\lambda + 1) \\
&= -(\lambda + 1)(\lambda(\lambda - 1) + 2) \\
&= -(\lambda + 1)(\lambda^2 - \lambda + 2) \\
&= -(\lambda + 1)^2(\lambda - 2)
\end{aligned}$$

Setting this equal to zero, we see the eigenvalues are $\lambda = -1, 2$. Next we find bases for the eigenspaces.

$\underline{E_{-1}(S)}$: We need to solve $(S + I)\mathbf{x} = \mathbf{0}$. Reducing $S + I$ yields

$$S + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_2 and x_3 are free variables and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence $E_{-1}(S)$ has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\underline{E_2(S)}$: We need to solve $(S - 2I)\mathbf{x} = \mathbf{0}$. Reducing $S - 2I$ yields

$$S - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x_3 is the free and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence $E_2(S)$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Now, in order to orthongonally diagonalize S, we need three orthonormal eigenvectors. We know that eigenvectors from distinct eigenspaces of S will be orthogonal. Hence the basis vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to the basis vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (which we can clearly see). However, the two basis vectors of $E_{-1}(S)$ are not orthogonal to each other. So we shouldn't take this basis; we need a different basis. We can do this using the Gram-Schmidt process!

The first vector is simply $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

The second is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

Now the three vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set of eigenvectors.

We now normalize them to make them orthonormal: $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$.

Now we have $A = QDQ^T$, where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

(b) Is S positive definite?

Since S has eigenvalue -1, it is neither positive definite, nor positive semi-definite.

(c) Redo part (b) without looking at the eigenvalues of S.

[Recall that S is positive definite if and only if $\mathbf{x}^T S \mathbf{x} > 0$.]

Since $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$, S is not positive definite.

5. Write a Julia function that takes in an $n \times n$ symmetric matrix and compute $\mathbf{e}_i^T S \mathbf{e}_i$ for all $1 \leq i \leq n$. The code should then either output "S is not positive definite," "S is not positive semi-definite", or "S may or may not be positive semi-definite." How can this code be made more efficient?

   First note that

   - if $\mathbf{e}_i^T S \mathbf{e}_i < 0$ for some i, then S is not positive semi-definite (and hence not positive definite either).
   - if $\mathbf{e}_i^T S \mathbf{e}_i \geq 0$ for all i and $\mathbf{e}_j^T S \mathbf{e}_j = 0$ for some j, then S is not positive definite.
   - if $\mathbf{e}_i^T S \mathbf{e}_i > 0$ for all i, then S may or may not be positive semi-definite (or positive definite).

   Hence we can write a for loop that checks these three cases.

   Since $\mathbf{e}_i^T S \mathbf{e}_i$ is simply the i-th entry of S, to make the code more efficient, simply determine the sign of the diagonal entries of S.